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GEOMETRIC MARGINAL ASYMMETRIC LAPLACE AND LINNIK DISTRIBUTION AND RELATED TIME SERIES MODEL

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ABSTRACT
Linnik distribution is heavy tailed compared to Laplace distribution and is used for modeling data sets in various fields especially in Finance and Economics. In this paper, a bivariate distribution related to geometric Pakes asymmetric Laplace and Linnik distribution is introduced and bivariate time series model corresponding to this distribution is developed.

KEYWORDS
Autoregressive process; Geometric marginal asymmetric Laplace and Linnik distribution; Geometric Pakes generalized asymmetric Linnik distribution; Geometric Stable distribution.

1. Introduction

In recent years there has been an increasing interest in developing the theory and applications of geometric stable distributions. The class of geometric stable distributions is a four-parameter family of distributions denoted by $GS_\alpha(\sigma, \beta, \mu)$ and conveniently described in terms of characteristic function

$$\Phi(t) = \frac{1}{1 + \sigma^\alpha |t|^\alpha w_{\alpha,\beta}(t) - i\mu t}$$

$$\text{where } w_{\alpha,\beta}(t) = \begin{cases} 1 - i\beta \, \text{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right) & \text{if } \alpha \neq 1 \\ 1 + i\beta \frac{2}{\pi} \, \text{sign}(t) \log|t| & \text{if } \alpha = 1. \end{cases}$$

The parameter $\alpha \in (0, 2]$ is the index of stability and determines the tail of the distribution. These classes of distributions arise as limiting distribution of geometric random sums of independent and identically distributed random variables. Since the geometric random sums frequently appear in many applied problems in

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various areas (see [3]), the geometric stable distributions have wide variety of applications especially in the field of reliability, biology, economics and financial mathematics.

When $\beta = 0, \mu = 0$, the geometric stable distribution has the characteristic function $\Phi(t) = \frac{1}{1+\sigma^\alpha |t|^\alpha}$, and the corresponding distribution is called Linnik distribution and is named after Yu.V.Linnik, who showed that the above function is a bonafide characteristic function of a symmetric distribution for any $0 < \alpha \leq 2$. The probability density function of the Linnik random variable when $\sigma = 1$ has the form

$$f_\alpha(x) = \frac{\sin \frac{\pi\alpha}{2}}{\pi} \int_0^\infty \frac{v^\alpha \exp(-v|x|)}{1 + v^{2\alpha} + 2v^\alpha \cos \frac{\pi\alpha}{2}} dv,$$

for $x > 0$ and for $x < 0$, $f_\alpha(x) = f_\alpha(-x)$. It may be noted that probability density and distribution functions of the Linnik random variable are not in closed form except for $\alpha = 2$, which corresponds to the Laplace distribution. The Laplace distribution is symmetric, and there were a number of asymmetric extensions in generalizing the Laplace distribution. [10] studied asymmetric Laplace (AL) distribution with characteristic function $\Phi(t) = \frac{1}{1+\sigma^2 t^2 - i\mu t}$, $\infty < \mu < \infty$, $\sigma \geq 0$, and discussed applications of it in the field of financial mathematics. [8] discussed a class of distributions, namely generalized asymmetric Laplace distributions, with characteristic function $\Phi(t) = (\frac{1}{1+\sigma^2 t^2 - i\mu t})^\tau$, $-\infty < \mu < \infty$, $\tau > 0$.

By specifying $\tau = 1, \sigma = 0$ and $\mu > 0$, we have an exponential distribution with mean μ and obtain symmetric Laplace distribution if $\tau = 1, \mu = 0$ and $\sigma \neq 0$. When $\sigma = 0$, the function is reduced to the characteristic function of a gamma variable with the scale parameter μ and the shape parameter τ.

[14] studied some asymmetric generalizations of geometric Linnik distribution. In Section 2, we present some aspects of Pakes generalized asymmetric Linnik distribution and geometric Pakes generalized asymmetric Linnik distribution. In Section 3, we introduce and study a bivariate distribution, namely geometric marginal asymmetric Laplace and asymmetric Linnik distributions and develop a bivariate time series model corresponding to this distribution. Conclusions are presented in Section 4.

2. Pakes generalized asymmetric Linnik distribution and geometric Pakes asymmetric Linnik distribution

[16] generalized the Linnik distribution and introduced a symmetric distribution, namely generalized Linnik distribution with characteristic function $\Phi(t) = (\frac{1}{1+\sigma^\alpha |t|^\alpha})^\tau$, $\sigma, \tau \geq 0, 0 < \alpha \leq 2$. It may be noted that when $\alpha = 2$ this reduces to the characteristic function of generalized Laplace distribution of [15]. Similar to generalized asymmetric Laplace distribution we can define an asymmetric distribution with characteristic function

$$\Phi(t) = (\frac{1}{1 + \sigma^\alpha |t|^\alpha - i\mu t})^\tau, -\infty < \mu < \infty, \sigma, \tau \geq 0, 0 < \alpha \leq 2. \quad (1)$$

This distribution is referred as the Pakes generalized asymmetric Linnik distribution and denoted by $PGAL_\alpha(\mu, \sigma, \tau)$. For properties of $PGAL_\alpha(\mu, \sigma, \tau)$, see [14].

When $\alpha = 2, \tau = 1$, it reduces to the asymmetric Laplace distribution of [10].

Theorem 2.1. *A $PGAL_\alpha(\mu, \sigma, \tau))$ random variable X with characteristic function (1) admits the representation $X \stackrel{d}{=} \mu W + \sigma W^{1/\alpha} Z$, where Z is symmetric stable with characteristic function $\Psi(t) = exp(-\sigma^\alpha |t|^\alpha)$ and W is a gamma random variable with probability density function $g(w) = \frac{1}{\Gamma(\tau)} w^{\tau-1} e^{-w}, w > 0, \tau > 0$, independent of Z.*

Proof. For Proof, see[14]. □

[17] introduced geometric exponential distribution and studied the properties of the renewal process with geometric exponential waiting time distribution. [6] and [7] studied geometric gamma and geometric Laplace distributions and developed autoregressive time series models. [4] introduced geometric Mittag-Leffler distribution and studied its properties including infinite divisibility and attraction to stable laws. [18] introduced and studied geometric Pakes generalized Linnik distribution and developed time series model using this distribution. A distribution with characteristic function

$$\Psi(t) = \frac{1}{1 + \tau log(1 + \sigma^\alpha |t|^\alpha - i\mu t)}, -\infty < \mu < \infty, \sigma, \tau \geq 0, 0 < \alpha \leq 2 \qquad (2)$$

is called geometric Pakes generalized asymmetric Linnik (GPGAL) distribution with parameters μ, σ, α and τ.

If X is a random variable with characteristic function (2), we represent it as $X \stackrel{d}{=} GPGAL_\alpha(\mu, \sigma, \tau)$. It may be noted that when $\tau = 1$ in (2), the corresponding distribution is the geometric version of asymmetric Linnik distribution and in such case we call it as geometric asymmetric Linnik distribution ($GPGAL_\alpha(\mu, \sigma, \tau = 1)$).

Theorem 2.2. *$GPGAL_\alpha(\mu, \sigma, \tau)$ distribution is the limit distribution of geometric sums of $PGAL_\alpha(\mu, \sigma, \tau)$ random variables.*

Theorem 2.3. *Let $\{X_n\}$ be a sequence of independent and identically distributed random variables and let N_p be a geometric random with mean $1/p$. Further, assume that N_p is independent of the $X_i's$. If $U_{N_p} = \sum_{i=1}^{N_p} X_i$ then the random variables U_{N_p} and X_i are identically distributed if Xi follows $GPGAL_\alpha(\mu, \sigma, \tau)$ distribution.*

Proof. For Proofs of Theorem 2.2 and 2.3, see[14]. □

3. Geometric marginal asymmetric Laplace and Linnik distribution

[9] introduced and studied a class of multivariate distributions called operator geometric stable laws by generalizing operator stable and geometric stable laws. As a particular case, they studied a new class of bivariate distributions namely marginal Laplace and Linnik distributions. [11] generalized this class of distributions and introduced and studied a class of bivariate distributions that contains marginal Laplace and Linnik distributions. The resulting class of bivariate distributions, namely generalized marginal asymmetric Laplace and asymmetric Linnik (GMALAL) distributions, has

the characteristic function

$$\Phi(t,s) = \left(\frac{1}{1+\sigma^2 t^2 + \eta^\alpha |t|^\alpha - i\mu t - i\nu s}\right)^\tau, \quad \sigma,\eta \geq 0, -\infty < \mu, v < \infty, \alpha \in (0,2], \tau \geq 0. \tag{3}$$

Let $\Psi(t,s)$ be the characteristic function of a geometrically infinitely divisible bivariate distribution given by the equation $\Phi(t,s) = \exp\left\{1 - \frac{1}{\Psi(t,s)}\right\}$, where $\Phi(t,s)$ is the characteristic function of an infinitely divisible bivariate distribution. Substituting (3) in the equation, we obtain

$$\Psi(t,s) = \frac{1}{1+\tau \log\left(1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - ivs\right)}$$
$$\sigma,\eta \geq 0, -\infty < \mu, v < \infty, \alpha \in (0,2], \tau \geq 0.$$

Hence

$$\Psi(t,s) = \frac{1}{1+\tau \log\left(1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - i\ v\ s\right)}$$

is the characteristic function of a geometrically infinitely divisible bivariate distribution.

A bivariate distribution with characteristic function

$$\Psi(t,s) = \frac{1}{1+\tau \log\left(1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - ivs\right)}, \quad \sigma,\eta \geq 0, -\infty < \mu, v < \infty, \alpha \in (0,2], \tau \geq 0 \tag{4}$$

is called geometric generalized marginal asymmetric Laplace and asymmetric Linnik (GGMALAL) distribution with parameters $\mu, \nu, \sigma, \eta, \alpha$ and τ .

If (X,Y) is a bivariate random vector with characteristic function (4), we represent it as $(X,Y) \stackrel{d}{=} GGMALAL_\alpha(\mu,\nu,\sigma,\eta,\tau)$.

Time series in which observations are of clearly non-Gaussian nature are very common in many areas. A number of literature have developed in recent years in modeling time series data with non-Gaussian, and more generally asymmetric marginal distributions. [2] discussed and studied conventional first order linear autoregressive model $X_n = \rho X_{n-1} + \epsilon_n$ with exponential marginal distribution. Subsequently, [12],[1] and [5] developed autoregressive models with different marginal distributions such as exponential, Laplace, and Mittag-Leffler distributions. These first order autoregressive models are developed using the self-decomposability property of the corresponding marginal distributions. [13] developed autoregressive models with type I and type II generalized geometric Linnik marginals.

Now it is useful to develop a bivariate time series model using GGMALAL marginal distributions. A one-parameter autoregressive model equivalent to TEAR (1) structure of [12] can be constructed corresponding to a set of bivariate time series data is as

follows:
Let $(\epsilon_n, \eta_n), n \geq 1$ be a sequence of independent and identically distributed bivariate random vectors and let $(X_0, Y_0) \stackrel{d}{=} GGMALAL_\alpha(\mu, \nu, \sigma, \eta, \tau)$ be a random vector with characteristic function (4). Define $\{(X_n, Y_n), n \geq 1\}$ as

$$X_n = \begin{cases} \epsilon_n \text{ w.p. } p \\ X_{n-1} + \epsilon_n \text{w.p. } (1-p) \end{cases} \quad \text{and } Y_n = \begin{cases} \eta_n & \text{w.p. } p \\ Y_{n-1} + \eta_n & \text{w.p. } (1-p) \end{cases} \tag{5}$$

where 'w.p.' stands for 'with probability' and $0 < p < 1$.
Let $\Phi_{(X_n,Y_n)}(t,s)$ and $\Phi_{(\varepsilon_n,\eta_n)}(t,s)$ be the characteristic function of (X_n, Y_n) and (ε_n, η_n) respectively. Then (5) gives

$$\Phi_{(\varepsilon_n,\eta_n)}(t,s) = \frac{\Phi_{(X_n,Y_n)}(t,s)}{p + (1-p)\Phi_{(X_{n-1},Y_{n-1})}(t,s)}. \tag{6}$$

If $\{(X_n, Y_n)\}$ be stationary sequence with $\text{GGMALAL}_\alpha(\mu, v, \sigma, \eta, \tau)$ marginal distribution, then from (6) we get

$$\Phi_{(\varepsilon_n,\eta_n)}(t,s) = \frac{1}{1 + p\tau \log\left(1 + \sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - ivs\right)}.$$

Hence,

$$(\varepsilon_n, \eta_n) \stackrel{d}{=} GGMALAL_\alpha(\mu, v, \sigma, \eta, p\tau). \tag{7}$$

Also it can be verified that, if $(X_0, Y_0) \stackrel{d}{=} \text{GGMALAL}_\alpha(\mu, v, \sigma, \eta, \tau)$ and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is an independent and identically distributed sequence of bivariate random variables given by (7), the first order autoregressive process (5) is stationary with $\text{GGMALAL}_\alpha(\mu, v, \sigma, \eta, \tau)$ marginal distribution.
Hence we have the following theorem.

Theorem 3.1. *Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$ be sequence of independent and identically distributed $GGMALAL_\alpha(\mu, v, \sigma, \eta, p\tau)$ random vectors and $(X_0, Y_0) \stackrel{d}{=} GGMALAL_\alpha(\mu, v, \sigma, \eta, \tau)$. Then equation (5) defines a stationary bivariate time series with GGMALAL marginal.*

We call the process (5) as first order GGMALAL autoregressive process (GGMALAL AR (1)).

4. Conclusion

Most of the data sets in the areas of financial mathematics, reliability, environmental studies etc, often do not follow the normal law but with asymmetric and heavy tailed character. A large number of research journals discussed applications of Laplace and asymmetric Laplace distributions in different fields where data exhibits asymmetric and heavy tailed character. In communication theory, frequently encountered impulsive noise possesses heavy tails, and so Laplace noise has been suggested as a best model. Also it is established that the Laplace distribution is considered as a model for the distribution of speech waves and the distribution is commonly encountered in image and speech compression. Empirical analysis of some important time series data, especially in the field of financial mathematics, environmental studies etc. shows that asymmetric and heavy tailed distributions, related to Laplace distribution, are more suitable for modeling the data. The applications of Laplace distribution in modeling sizes of sand particles, diamonds etc. are also well established in many research articles (for more details see Kotz et al. (2001)). In this paper, we examined the distributions related to Laplace and asymmetric Laplace distributions and can be used as an appropriate model in the areas where Laplace and Linnik distributions do not provide a better fit. Much will be learned in future by further analysis related to the applications of this new class of distributions across a range of contexts.

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